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SUBJECT: Single-Pass Orbit Determination by Smoothing Radar Orbital Tracking Data - Case 20061

DATE: February 27, 1964

FROM: G. H. Myers

MEMORANDUM FOR FILE

I. Introduction

This memorandum will discuss the problem of computing the smoothed position and velocity of an orbiting body as it passes over a single radar tracking station. This problem is different from the usual orbit determination problem, in which an object is tracked by many stations and a "best fit" orbit is made to the data, usually by a form of differential correction scheme (Ref. 1). However, differential correction methods are often not particularly efficient or accurate for single-pass data (Ref. 1). The procedure described here is most effective for the case of the single pass, but is not particularly well adapted to reducing data from a number of stations.

An estimate of the accuracy obtainable for the Apollo parking orbit is shown in Figure 1, which gives velocity error as a function of tracking time, for different ranges to the satellite. The assumed radar errors are tabulated on the Figure As is typical of smoothing techniques, there is an optimum smoothing time which results in minimum error, and using more or less data at one time than this optimum results in less accuracy. For most cases of interest, the object being tracked is not above the horizon for the full optimum period. The best policy then is to track for as long as possible. Using smoothed data of this sort would be most effective if it were desirable to obtain a quick estimate of an orbit from an isolated tracking station, such as a tracking ship. Relatively small facilities could be used for the computations required, and then only either a set of orbital elements or the position and velocity vector at one time could be transmitted to a central station, where the orbit would be computed.

The basic principle used in the method to be described is to use measured position values to compute the gravitational acceleration. The computed accelerations are then integrated in an inertial coordinate system and subtracted from measured values. The differences are smoothed in polynominal filters, and the integrated quantities are then added in to give the total smoothed velocity or position.

Errors in smoothing arise from radar noise and radar bias, but bias errors enter the smoothed data in two different ways. Certain bias errors (azimuth and site location, in particular) correspond basically to a rotation or translation of coordinates. Thus velocities will in general be computed correctly with respect to the radar if such biases are present, but directions and locations will be in error by the amount the radar is in error. It is apparent that measurements from a single radar only cannot remove errors of this sort. Other bias effects (mainly in range and elevation) produce erroneous accelerations in the measured quantities. These accelerations are detected by the smoothing filters and their effects are usually exaggerated. The bias-caused accelerations produce a dynamic error at the output of the filter, which tends to increase with increasing smoothing time, while the noise error tends to decrease with increasing smoothing time. Thus there is some time for which the total error is minimized.

The actual dynamic errors for any pass depend to a great degree on the geometry of that particular pass. For example, while the gravitational acceleration is almost constant, it would affect mainly angle measurements for a pass low on the horizon, but mainly range measurements for a pass nearly overhead. The problem is further complicated by the fact that most radars are considerably more accurate in range than in angle. In the error analysis, some assumptions are made to obtain results of reasonable generality, usually in the nature of attempting to select a situation whose geometry presents a worst case in a particular coordinate, and then computing the dynamic errors in that coordinate.

II. General Procedure

The basic idea of this smoothing procedure, as discussed in the previous section, is to remove nominal effects by using gravitational accelerations computed from the measured positions. The procedure, which is also diagrammed in Figure 2, is described below.

1. Convert the radar tracking data (assumed to be range, azimuth, and elevation) into an earth-centered, inertial coordinate system.

2. Calculate the direction and magnitude of the gravitational acceleration at each point from

$$\bar{a} = - \frac{k}{|\bar{r}|^3} \bar{r} \qquad (1)$$

where $\bar{r}$ is the radius vector from the center of the earth to the point in question and k is the gravitational constant. The unsmoothed radar data is to be used to compute $\bar{r}$.

3. Integrate $\bar{a}$ numerically in the inertial coordinate system to get the nominal increment in velocity from the beginning of the pass.

4. Subtract the computed velocity increment from first differences in the measured position data.

5. Smooth the resulting differences by means of growing velocity filters (also called 1,1 filters) as described in

Reference 2 and in the Appendix. These filters require a minimum of data storage in the computer, and are easy to implement. The final velocity is the filter output plus the computed integral of gravity (in each coordinate).

6. If it is desired to smooth position, the acceleration may now be integrated a second time, subtracted from the position (the position should be delayed by one cycle to allow for delays in the integration) and smoothed in a 0,0 filter (Appendix and Ref. 2). At the output of the filter, the integrated accelerations are added along with the output of the 1,1 filter multiplied by $1/2(T-\Delta t)$, where T is the smoothing time and Δt is the interval between measurements. For most radars, the bias errors are large enough to make it unprofitable to smooth position, because most of the bias error is not removable by smoothing a single pass position. Position smoothing will not be discussed in any detail in this memorandum.

III. Error Components

Numerous approximations are made in this section to simplify computing the errors, but it is not intended that the actual operational procedure described in the previous section have the same omissions. For example, the effects of earth's rotation will be neglected here since it does not affect the magnitude of the smoothing errors. It must be included in all of the coordinate rotations actually performed, however, or else a very large error will be incurred.

Most radars measure in the familiar range-azimuth-elevation coordinate system. However, it will be simpler if it is assumed that the radar measures positions in standard spherical coordinates, illustrated in Figure 5, where $\theta = -A$; and $\varphi = 90° - E$. A lower case "r" will be used for range to be consistent with standard notation for spherical coordinates. The inertial coordinate system assumed will be the x-y-z coordinates shown in Figure 5 translated to the center of the earth. The rotation of the earth, which is a very small effect for the length of time a satellite is in view of a station, will be neglected in the following analysis (although it must be considered in actually smoothing the data). The coordinate system of Figure 5 is therefore also an inertial coordinate system, which is completely equivalent to the earth-centered system for velocities and accelerations (since the two coordinate systems differ only by translations). Position errors in the radar-centered coordinate system are identical to the errors in the earth-centered system. In general, no distinction will be made between these two coordinate systems unless only one of them can be used, such as in computing gravitational accelerations. If a coordinate is earth-centered, the subscript "c" will be used.

The rectangular coordinates are related to the radar coordinates by the familiar relations

$$x = r \cos \theta \sin \varphi$$
$$y = r \sin \theta \sin \varphi \qquad (2)$$
$$z = r \cos \varphi$$

The position vector in rectangular coordinates will be called $\overline{R}$, while the position vector in radar coordinates will be $\overline{R}_R$. Thus,

$$\overline{R} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; \qquad \overline{R}_R = \begin{bmatrix} r \\ \theta \\ \varphi \end{bmatrix} \qquad (3)$$

The same notation will be used for velocities: $\dot{\overline{R}} = V$, and $\mathring{\overline{R}}_R = \overline{V}_R$. We then have by differentiation of (2)

$$\overline{V} = A\ \overline{V}_R \qquad (4)$$

where A is the matrix

$$A = \begin{bmatrix} \cos \theta \sin \varphi & -r \sin \theta \sin \varphi & r \cos \theta \cos \varphi \\ \sin \theta \sin \varphi & r \cos \theta \sin \varphi & r \sin \theta \cos \varphi \\ \cos \varphi & 0 & -r \sin \varphi \end{bmatrix} \qquad (5)$$

The accelerations in rectangular and radar coordinates are related by

$$\overline{a} = \dot{A}\ \overline{V}_R + A\ \overline{a}_R \qquad (6)$$

where the matrix $\dot{A}$ is the derivative of (5) given by

$$\dot{A} = \begin{bmatrix} -\dot{\theta} \sin\theta \sin\varphi & -\dot{r} \sin\theta \sin\varphi & \dot{r} \cos\theta \cos\varphi \\ +\dot{\varphi} \cos\theta \cos\varphi & -r\dot{\theta} \cos\theta \sin\varphi & -r\dot{\theta} \sin\theta \cos\varphi \\ & -r\dot{\varphi} \sin\theta \cos\varphi & -r\dot{\varphi} \cos\theta \sin\varphi \\ \dot{\theta} \cos\theta \sin\varphi & \dot{r} \cos\theta \sin\varphi & \dot{r} \sin\theta \cos\varphi \\ +\dot{\varphi} \sin\theta \cos\varphi & -r\dot{\theta} \sin\theta \sin\varphi & +r\dot{\theta} \cos\theta \cos\varphi \\ & +r\dot{\varphi} \cos\theta \cos\varphi & -r\dot{\varphi} \sin\theta \sin\varphi \\ -\dot{\varphi} \sin\varphi & 0 & -\dot{r} \sin\varphi \\ & & -r\dot{\varphi} \cos\varphi \end{bmatrix} \tag{7}$$

It is now possible to express mathematically the steps described in the preceding section. In the ideal case, the radar measures a sequence of positions, $\overline{R}_R(t) + \overline{N}_R$, where $\overline{N}_R$ represents the noise in each coordinate. These positions are transformed to rectangular, earth-centered coordinates $\overline{R}_c$, and the gravitational acceleration is computed from Equation (1). The integral of this acceleration, which will be called $\overline{V}_a$, is subtracted from first differences in $\overline{R}$, and the result is smoothed. The "gravitational velocity" $\overline{V}_a$ is then added back in, giving the true velocity plus an error due to the noise, which may be computed using the standard filter formulas. There is no dynamic error in the procedure in the ideal case.

Now consider the effect of biases. Suppose the radar coordinates as measured are a quantity $\overline{R}_R' = \overline{R}_R+\overline{N}_R+\overline{B}_R$, where the vector $\overline{B}_R$ is given by

$$\overline{B}_R = \begin{bmatrix} \Delta r \\ \Delta\theta \\ \Delta\varphi \end{bmatrix}$$

and Δr, $\Delta\theta$, and $\Delta\varphi$ are unknown biases independent of time. The position in rectangular coordinates is now

$$\overline{R}' = \overline{R} + A(\overline{N}_R + \overline{B}_R) \tag{9}$$

where $\overline{R}$ is the true position in rectangular coordinates as a function of time. The "A" matrix transforms both noise and bias to rectangular coordinates if it is assumed that these small errors are approximately equal to the differentials of position. The gravitational acceleration is now computed as a function of $\overline{R}_c'$ (which is just $\overline{R}'$ translated to the center of the earth if earth's rotation is neglected), so

$$\overline{V}_a' = \int \overline{a}(\overline{R}')\ dt \tag{10}$$

as contrasted to the true component

$$\overline{V}_a = \int \overline{a}(\overline{R})\ dt \tag{11}$$

The velocities computed with the biased data are

$$\overline{V}' = \overline{V} + \dot{A}\ \overline{B}_R + \overline{V}_N \tag{12}$$

where $\overline{V}_N$ are the components due to noise. The noise part, $\overline{V}_N$, will be dropped for the moment and considered later, since methods for computing the error due to noise are fairly standard. If only dynamic components are considered, Equation (12) can be rewritten as

$$\overline{V}' = \overline{V}_o + \overline{V}_a + \dot{A}\,\overline{B}_R\,, \tag{13}$$

where $\overline{V}_o$ is the initial velocity. The quantity to be smoothed, which will be called $\overline{V}_d$, has Equation (10) removed from it, giving

$$\overline{V}_d = \overline{V}_o + \left(\overline{V}_a - \overline{V}_a'\right) + \dot{A}\,\overline{B}_R \tag{14}$$

Ideally, of course, $\overline{V}_d$ is just equal to the constant velocity $\overline{V}_o$. If each component of $\overline{V}_d$ is now passed through identical 1,1 filters, the output is (Ref. 2)

$$\overline{V}_{df} = \overline{V}_d + \frac{T}{2}\frac{d}{dt}\left(\overline{V}_d\right) = \overline{V}_d + \frac{T}{2}\left[\overline{a}(\overline{R}) - \overline{a}(\overline{R}') + \ddot{A}\,\overline{B}_R\right] \tag{15}$$

where $\overline{V}_{df}$ denotes the filter output, and T is equal to the smoothing time. After the "gravitational velocity" is added back in, the output of the filter (and the final answer), $\overline{V}_f$, is

$$\overline{V}_f = \overline{V}_o + \overline{V}_a + \overline{E}_{VD}\,, \tag{16}$$

where $\overline{E}_{VD}$ is the dynamic error in velocity, given by

$$\overline{E}_{VD} = \dot{A}\,\overline{B}_R + \frac{T}{2}\left[\overline{a}(\overline{R}) - \overline{a}(\overline{R}') + \ddot{A}\,\overline{B}_R\right] \qquad (17)$$

The noise error (which was neglected in the previous paragraphs) is given by

$$\sigma_{VN}^2 = \frac{6}{T^3 f_c}\,\sigma_N^2\ , \qquad (18)$$

where σ_{VN}^2 is the variance of the output error in each coordinate and σ_N^2 is the variance of the input in each coordinate (after transformation to rectangular coordinates), assuming (as is customary) that the noise in range, angle, and elevation are independent, normal, uncorrelated sources with Markoff distributions having a corner frequency f_c. The smoothing time is T in Equation (18).

Before finding the optimum filter, it is necessary to estimate the dynamic error given in Equation (17). Each of the three terms in that equation will be determined separately in the following paragraphs. It should be noted that, because the first term in (17) is not multiplied by the smoothing time of the filter, that term plays no part in the filter optimization, but is just an error added on to the error in the filter output. The three parts of (17) will be evaluated in the order in which they appear in that equation, starting with the $\dot{A}\,\overline{B}_R$ term.

If the various elements of the matrix A are compared with the coordinate transformations of Equation (2), it may be seen that an equivalent way of writing A is

$$A = \begin{bmatrix} x/r & -y & x(\varphi+90°) \\ y/r & x & y(\varphi+90°) \\ z/r & 0 & z(\varphi+90°) \end{bmatrix} \qquad (19)$$

Similarly, $\dot{A}$ may be written as

$$\dot{A} = \begin{bmatrix} \dot{x}/r-x\dot{r}/r^2 & -\dot{y} & \dot{x}(\varphi+90°) \\ \dot{y}/r-y\dot{r}/r^2 & \dot{x} & \dot{y}(\varphi+90°) \\ \dot{z}/r-z\dot{r}/r^2 & 0 & \dot{z}(\varphi+90°) \end{bmatrix} \qquad (20)$$

(The symmetry becomes more apparent if $-y$ is written as $x(\theta+90°)$ and x is written as $y(\theta+90°)$.) The error in $\overline{R}$ in vector form is

$$A\ \overline{B}_R = \overline{u}_r \Delta r + \overline{u}_\theta r \sin\varphi\ \Delta\theta + \overline{u}_\varphi r\ \Delta\varphi\ , \qquad (21)$$

where $\overline{u}_r$, $\overline{u}_\theta$, and $\overline{u}_\varphi$ are unit vectors in the r, θ, and φ directions respectively. An examination of the elements of (20) reveals the following facts:

1. Azimuth biases (determined by the second column) correspond to a rotation of coordinates by the amount of the bias. The maximum error caused by an azimuth bias in the $\dot{A}\ \overline{B}_R$ term is is therefore $V_h \Delta\theta$, where V_h is the horizontal component of velocity. The maximum is attained if V_h is in either the x or y directions.

2. Elevation biases (determined by the third column) are similar in nature to a coordinate rotation, but the rotated coordinate system keeps changing. However, since the three elements in the third column of (20) are the three components of velocity in an orthogonal coordinate system, the maximum error caused by an elevation bias in the $\dot{A}\,\overline{B}_R$ term is $V\Delta\varphi$, where V is the magnitude of the total velocity. This maximum would occur if the total velocity were in the direction of one of the rotated axes of the third column of (20).

3. The range situation is considerably more complicated, but fortunately the range terms in $\dot{A}\,\overline{B}_R$ are negligible, for practical radars. The maximum value that any of the terms of the first column of (20) can have is $2V/r$, which would occur if the object being tracked was flying along one of the coordinate axes directly towards the radar. The total bias error is then $2V\Delta r/r$. If the minimum value of r is 100 miles, and the bias corresponds to 10 feet, then $2\Delta r/r$ is at least 3.3×10^{-5}, while the angle biases are of the order of 10^{-4}.

In most actual cases of tracking an object in earth orbit, the vertical velocity ($\dot{z}$) is quite low compared to the horizontal velocity, so that $V_h \approx V$. We will therefore assume that the maximum value of $\dot{A}\,\overline{B}_R$ in any coordinate is $V\sqrt{\Delta\varphi^2 + \Delta\theta^2}$, which is a slightly pessimistic estimate.

This component of error will be called the "velocity rotation error", and will be denoted by ε_{VR}.

Evaluation of the difference $a(\overline{R}) - a(\overline{R}')$ is somewhat more straightforward. Recaling (8), $\overline{R}' = \overline{R} + \overline{B}_R$. Replacing differentials by derivatives gives

$$\Delta\overline{a} = A(R) - a(R') = \sum_j \frac{\partial \overline{a}}{\partial R_j} \Delta R_j = C\Delta\overline{R} = C\ A\ \overline{B}_R \qquad (22)$$

where the elements of C are given by

$$c_{ij} = \frac{\partial a_i}{\partial R_j} = \begin{cases} \dfrac{3kR_{ci}R_{cj}}{|\overline{R}_c|^5}, & i \neq j \\[2ex] \dfrac{3kR_{ci}^2}{|\overline{R}_c|^5} - \dfrac{k}{|\overline{R}_c|^3}, & i = j \end{cases} \qquad (23)$$

and the subscripts "c" indicate that the coordinates must be measured from the center of the earth. The differentials of position, however, are the same for coordinates measured from the center of the earth or from the radar, since the two coordinate systems are parallel (because the rotation of the earth was neglected). (Even if the rotation were considered however, the effect would be negligible, since the earth rotates only 1/4 of a degree per minute.) The largest error in any one coordinate occurs if one of the axes is in the direction of the radius vector. If we assume all of the bias error from either azimuth or elevation is in the same coordinate direction (the range error is negligible), then the error due to this source is

$$E_a = \frac{2k}{|\bar{R}_c|^3} [r\Delta(\text{angle})] \tag{24}$$

where Δ(angle) is the bias in azimuth or elevation. Since these two biases are equal for most radars, the distinction is not important. In (23), "r" is measured from the radar. Equation (23) is also approximately equal to (for low earth orbits)

$$E_a = \frac{2gr\Delta(\text{Angle})}{|\bar{R}_c|} \tag{25}$$

where "g" is the acceleration of gravity at the earth's furface. This component will be called "acceleration error", denoted by E_a.

The term $\ddot{A}\ \bar{B}_R$ may be evaluated relatively easily. The derivatives of the second and third columns of A, from equation (20), are obviously the components of acceleration in rectangular coordinates, and are therefore limited to the value "g". The errors due to angle biases are therefore just "g" times the bias error. It is easier to evaluate the range term by examining (21), where it may be seen that the range component of $\ddot{A}\,\overline{B}_R$ is equal to

$$\frac{d^2}{dt^2}\,\overline{u}_r \Delta r = \left(\frac{V^2}{r}\,\overline{u}_r + r\ddot{\psi}\,\overline{u}_\psi\right)\frac{\Delta r}{r} \tag{26}$$

where ψ is the angle in the plane containing the orbit and the tracker, and $\overline{u}_\psi$ is the corresponding unit vector. For all practical purposes, this has the value V^2/r, since the second term in (26) has "g" as its maximum value. The total error due to this component is therefore

$$\ddot{A}\,\overline{B}_R = \left(\frac{V^2}{r}\,\frac{\Delta r}{r}\right) + g\,(\Delta\theta^2 + \Delta\varphi^2)^{1/2} \tag{27}$$

This component will be called the acceleration rotation error, ε_{aR}.

IV. Filter Optimization

Formulas for the noise error and the dynamic error were computed in the previous section. If $\Delta\psi = \sqrt{\Delta\theta^2 + \Delta\varphi^2}$ rms = total angle bias error, and σ_ψ is its standard deviation, then the dynamic and noise errors computed may be summarized in the following equations:

$$\varepsilon_{VR} = V\Delta\psi$$

$$\varepsilon_a = \frac{2gr\Delta\psi}{|\bar{R}_c|}$$

$$\varepsilon_{aR} = \left(\frac{V^2}{r}\,\frac{\Delta r}{r}\right) + g\Delta\psi \tag{28}$$

$$\sigma_N = r\sigma_\psi$$

The total error is given by equation (17). If (17) is differentiated to determine the value of smoothing time T which gives minimum error, the result is found to be

$$T_{opt} = 2.36(\Delta t)^{1/5}(\sigma_N/\sigma_D)^{2/5} \tag{29}$$

where σ_D is the rms dynamic error. This equation is plotted in Figure 3. The variance at the minimum is therefore

$$\sigma^2_{opt} = V^2\sigma^2_\psi + 2.30(\Delta t)^{2/5}\ \sigma_D^{6/5}\ \sigma_N^{4/5} \tag{30}$$

The right-hand part of this equation (that is, the error excluding the velocity rotation error) is plotted in Figure 4. It is not practical to include the velocity rotation error in the same curve as the rest of the error, because the rotation errors depends only on angle error, while the remainder of the dynamic error is a combination of both angle and range error. Figure 1 gives the total error as a function of smoothing time for a case of particular interest to the Apollo program. The error sources are listed in Table 1. The velocity rotation error has been included in the total error on an rms basis. While this procedure is not exactly correct, since part of the remaining error is due to angles and is correlated with the velocity rotation error, this procedure seems the most reasonable one to follow. In any event, the velocity rotation error by far dominates the net error. For the cases considered in Figure 1, this component amounts to 4.2 feet per second.

In order to use the formulas developed here, it is necessary to assume that the object being tracked stays at some "average" distance from the radar, and Figure 1 is plotted in the cases when this distance is one hundred and five hundred miles. It may be observed that the error does not depend very strongly on this assumed average slant range, and sufficient accuracy is probably achieved if the slant range at closest approach is used (which will tend to give a pessimistic estimate of dynamic error and an optimistic estimate of noise error). In any application, of course, there is usually no need to include data after the optimum smoothing time has been reached, but since the error does not increase very rapidly after this time, from an operational standpoint it may be desirable to include all of the tracking data. The degradation is almost negligible, and the operational procedures would then not have to depend on the geometry of a particular pass.

V. System Applications

In the introduction it was pointed out that the procedures described in this memorandum are well-suited to determining orbits at an isolated station such as a tracking ship. A small digital computer with a modest memory (or even possibly an analog computer) could easily perform the computations indicated, and compute the orbital elements from the positions and velocities. The six elements and a time reference could then be transmitted to a central point over a communications link with a very low capacity compared to the capacity required to transmit several minutes of tracking data. This technique could also be used profitably to obtain an initial estimate of the orbit, using data obtained at the first tracking station. This estimate, plus an estimate of the covariance matrix, can then be used as the initial state of a standard orbit--determination program. Comparison of the accuracies obtained in Figure 1 with those obtainable by a standard program for the same amount of tracking (Ref. 3) show that the polynominal smoothing technique gives 20 to 30 per cent more accuracy than a differential-correction method. However, the differential-correction method uses large amounts of data more efficiently than the polynominal method. Thus it seems reasonable to use polynominal smoothing to measure small arcs of orbits, and differential-correction methods to measure large areas.

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WH-4325-GHM-GMR G. H. MYERS

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Appendix

Growing Polynomial Filters

The "memory" of a filter is the time span during which input data affects the output. Thus if the memory of a filter is T seconds, all input data collected during the previous T seconds (and only that data) affect the output. The memory of a "growing" filter increases with time. Thus if t_0 is the time at which filtering action started, and t is the present time, then the smoothing time is $T = (t-t_0)$. The "0,0" filter discussed in the text makes a least-squares fit of a constant to position data perturbed by white noise, where the fit extends over the smoothing time of the filter. The "1,1" filter makes a least-squares fit of a constant to first differences of such position data. Thus the 0,0 filter measures position and the 1,1 filter measure velocity.

The most attractive operational features of growing filters are their simplicity and economical use of computer storage. In particular, they do not require storage of all the data points for the past T seconds; they preserve this data by storing a few appropriate quantities. The formulas for these filters are below.

The notation below is that measurements of position x_k are made Δt seconds apart. The input to the filter is μ_k, and the (smoothed) output is v_n. The subscript "k" specifies the input data point, so $-\infty < k < \infty$. The subscript "n" is referenced to the start of the filter, so $0 < n < T/\Delta t$. That is, n is zero when the filter is "turned on", and is always equal (in terms of data points) to the memory of the filter. The quantity "α" is the result of an intermediate calculation.

A. 0,0 filter

$$\mu_k = x_k$$

$$v_n = v_{n-1} + \frac{1}{n+1} (\mu_n - v_{n-1})$$

B. 1,1 filter

$$\mu_k = \frac{\Delta x_k}{\Delta t}$$

$$\alpha_n = \alpha_{n-1} + \frac{2}{n+1} (\mu_n - \alpha_{n-1})$$

$$v_n = v_{n-1} + \frac{3}{n+2} (\alpha_n - v_{n-1})$$

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2. Blackman, R. B., "Variable Smoothing of Discrete Data by Cascaded Simple Recursive Steps" - MM-58-121-16; December 1, 1958.
3. Priester, R. W., and Replogle, J. O., "Apollo Communications and Tracking System Performance During Earth Parking Orbit - Case 20061" - Memorandum for File, February 17, 1964.

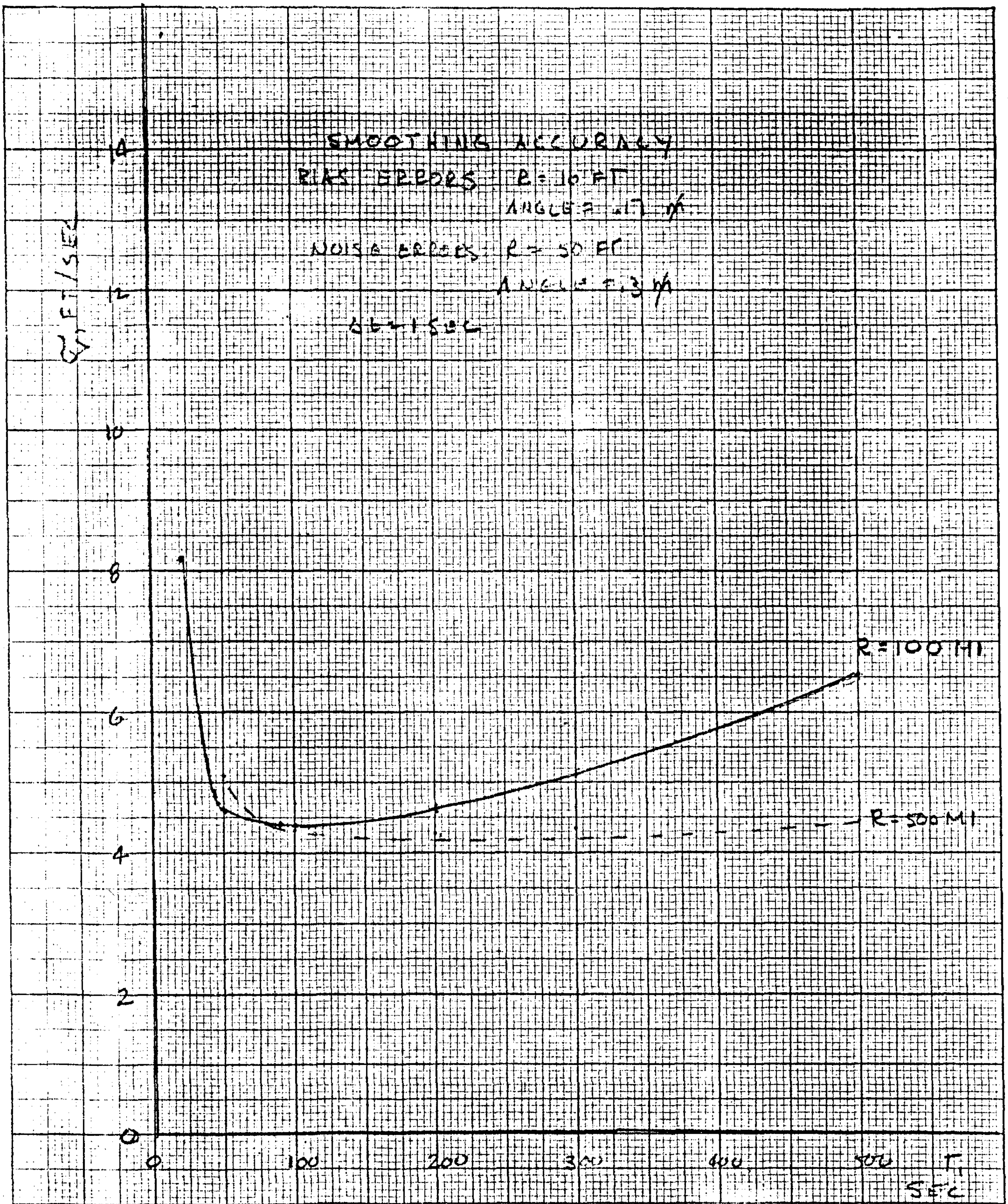
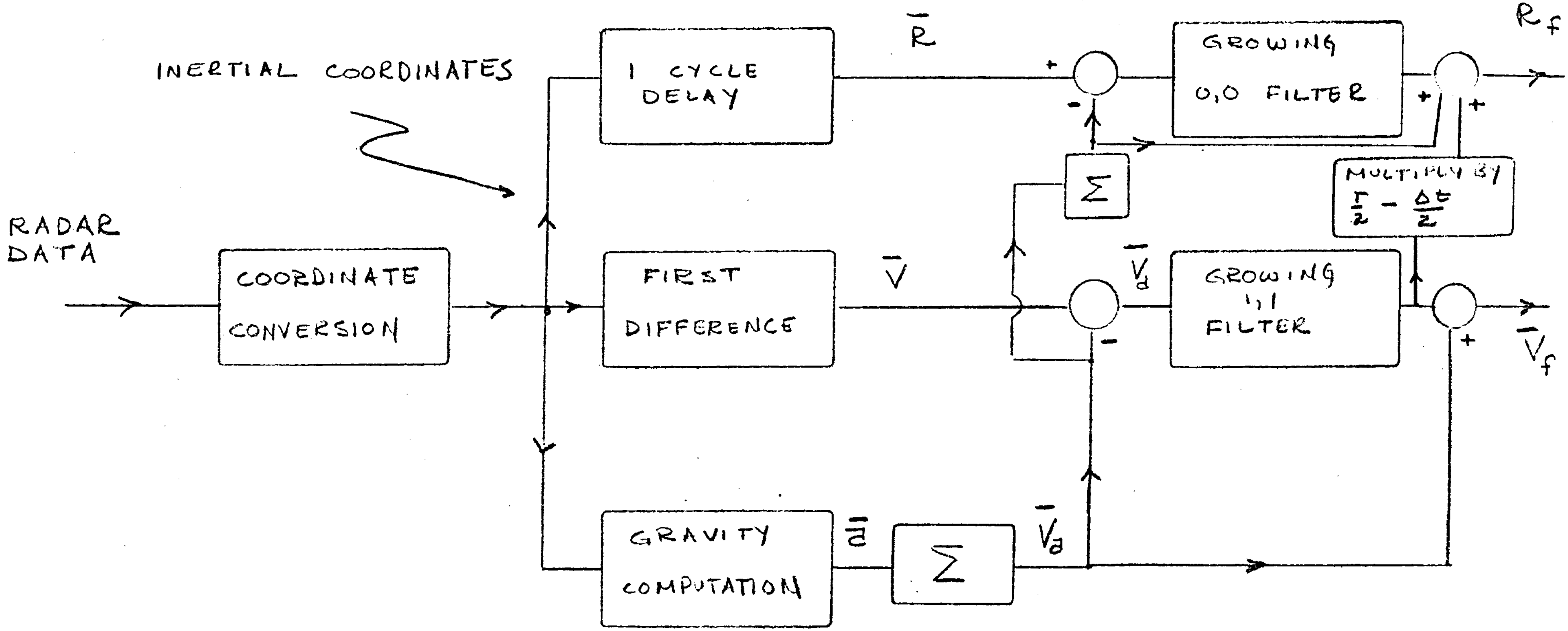
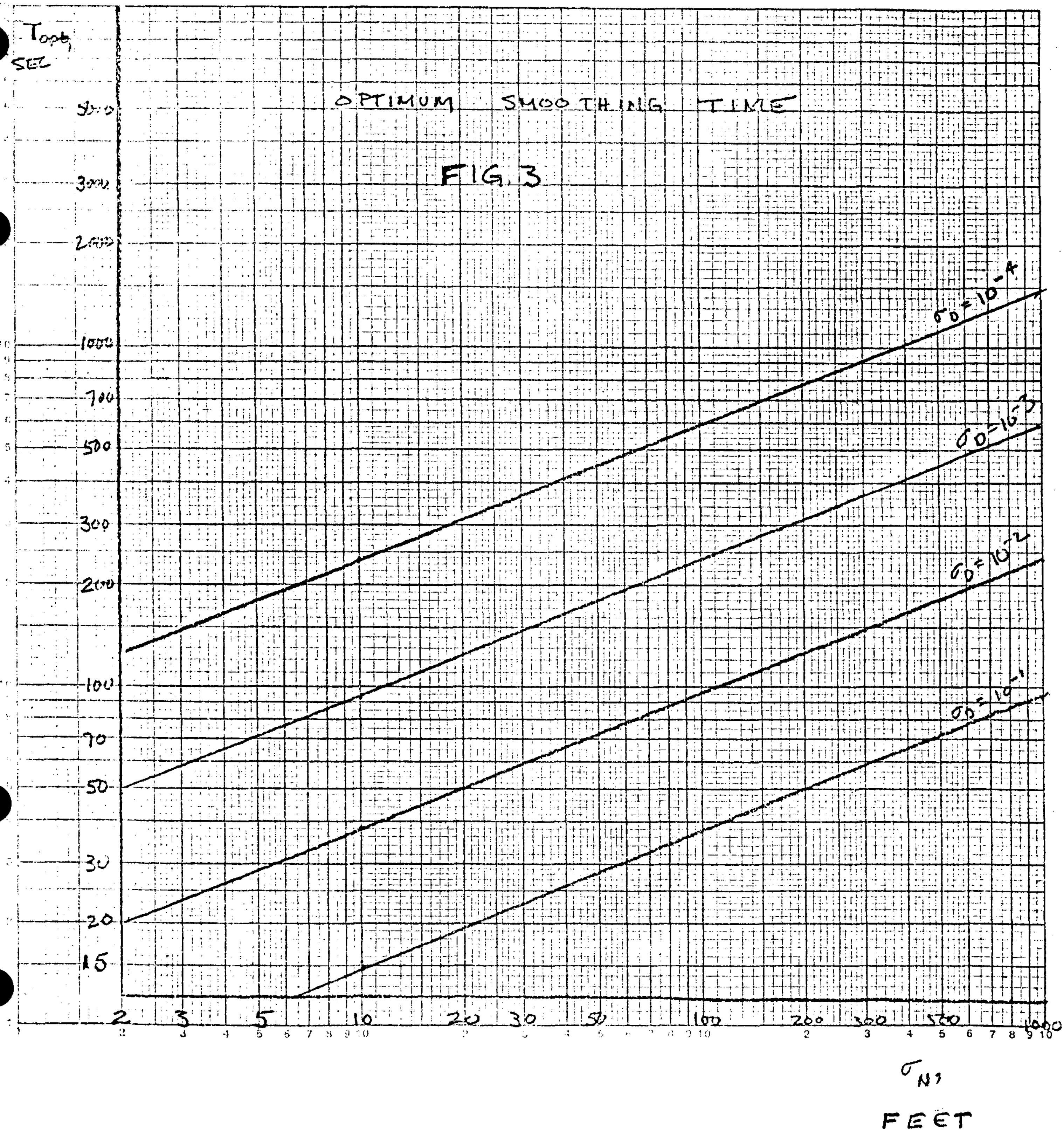

SMOOTHING ACCURACY
BIAS ERRORS: R = 16 FT
ANGLE = .17 M
NOISE ERRORS: R = 50 FT
ANGLE = .3 M
Δt = 1 SEC
FT/SEC
14
12
10
8
6
4
2
0
R = 100 MI
R = 500 MI
0
100
200
300
400
500
T, SEC

FIG. 1

FIGURE 2

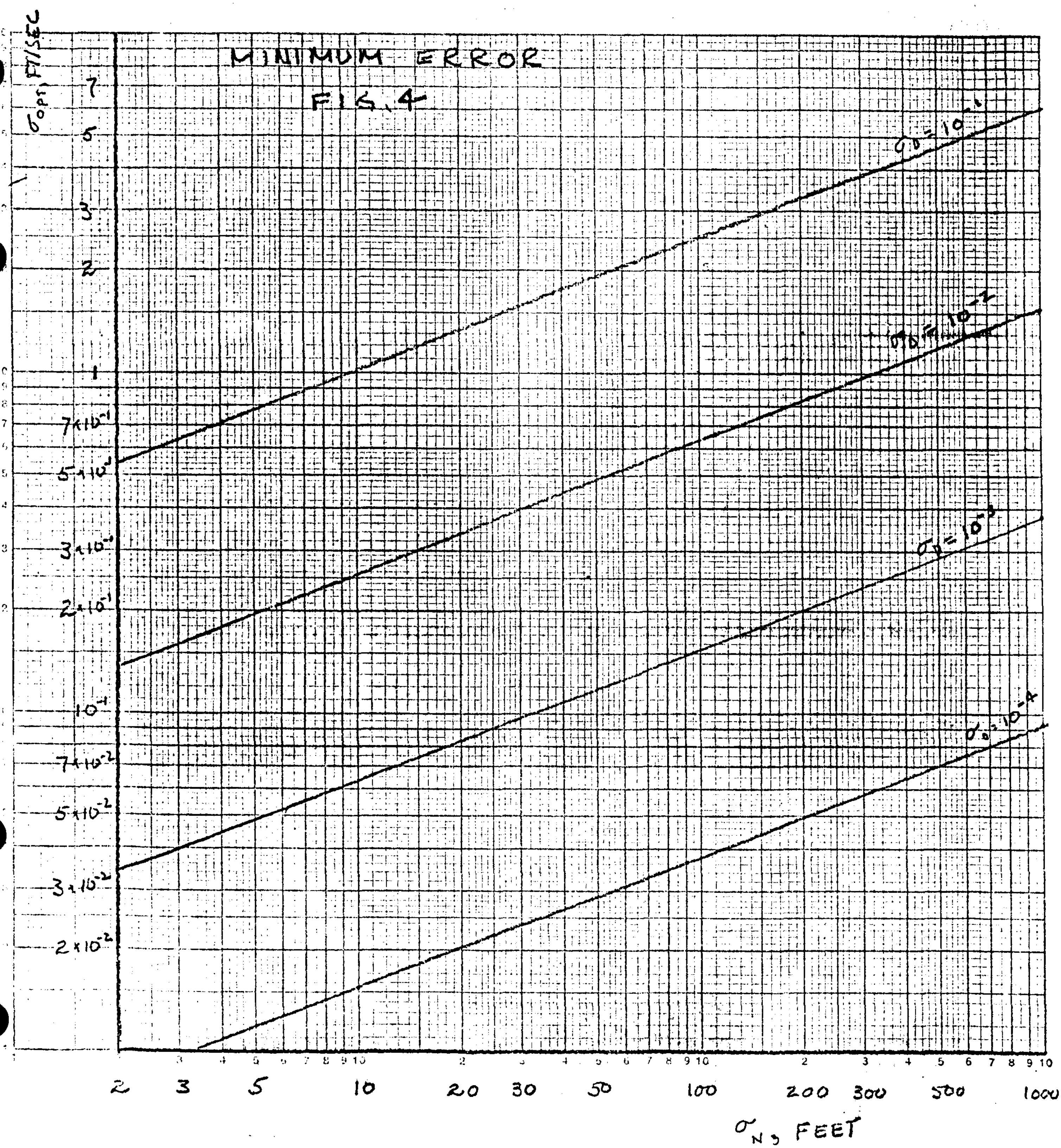
DIAGRAM OF SMOOTHING PROCESS





OPTIMUM SMOOTHING TIME
T_{opt}, SEC
$\sigma_D = 10^{-4}$
$\sigma_D = 10^{-3}$
$\sigma_D = 10^{-2}$
$\sigma_D = 10^{-1}$
σ_N, FEET

FIG. 3


MINIMUM ERROR
FIG. 4
σ_{opt}, FT/SEC
$\sigma_D = 10^{-1}$
$\sigma_D = 10^{-2}$
$\sigma_D = 10^{-3}$
$\sigma_D = 10^{-4}$
1
7×10^{-1}
5×10^{-1}
3×10^{-1}
2×10^{-1}
10^{-1}
7×10^{-2}
5×10^{-2}
3×10^{-2}
2×10^{-2}
2 3 5 10 20 30 50 100 200 300 500 1000
σ_N, FEET

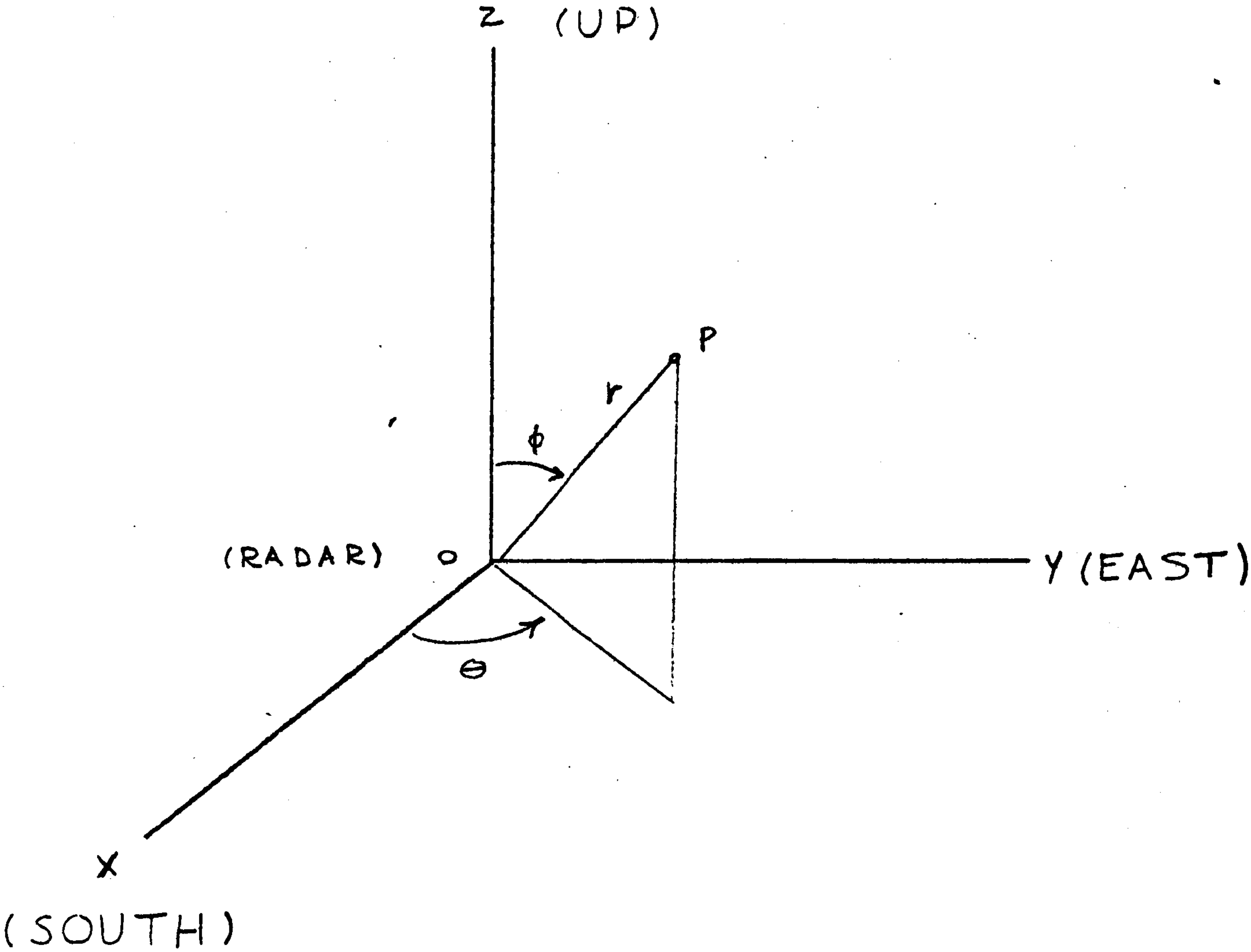


FIG. 5